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Conformal covariance in the framework of Wilson's renormalization group approach

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Abstract. We construct a conformal operator in analogy to the generating operator of Wilson's incomplete-integration renormalization group. The invariance of the partition function with respect to that conformal operation yields identities among the cumulants. Evaluating these identities we find a generalized and corrected form of the selection rule which determines those two-point cumulants which show a long-range tail. A general equation which governs the asymptotic form of the three-point cumulants is established. It is solved for several examples which involve operators of vector- or tensor-type. It is found that surface effects cannot be excluded *a priori*. However, the asymptotic expressions for the cumulants are consistent with a neglect of surface effects.

1. Introduction

Near the critical temperature of a second-order phase transition, the fluctuations of local quantities become correlated over distances which are large on a microscopic scale. This property is explained by the renormalization group approach, which shows that the cumulants $\langle \pi_i A_i(s\rho_i)\rangle_c$ of fluctuating quantities ('operators') $A_i(\rho_i)$, which are localized at the points ρ_i, behave asymptotically as

$$\left\langle \prod_{i=1}^{n} A_i(s\rho_i)\right\rangle_c \underset{s\to\infty}{=} C_{1,\ldots,n}(\rho_1,\ldots,\rho_n)s^{\sum_{i=1}^{n} x_i}. \tag{1.1}$$

The critical exponent x_i of the operator $A_i(\rho)$ can be calculated within the renormalization group approach. The physical idea underlying that theory can be formulated as dilatation covariance of the system.

The coefficient $C_{1,\ldots,n}(\rho_1,\ldots,\rho_n)$ can be studied by exploiting conformal covariance. Using the framework of field theory, Polyakoff (1970) has argued that for rotational invariant operators, conformal covariance fixes the form of the three-point cumulant:

$$C_{1,2,3}(\rho_1,\rho_2,\rho_3)=\Gamma_{1,2,3}|\rho_1-\rho_2|^{x_1+x_2-x_3}|\rho_2-\rho_3|^{x_2+x_3-x_1}|\rho_3-\rho_1|^{x_3+x_1-x_2}. \tag{1.2}$$

Here $\Gamma_{1,2,3}$ denotes a constant. For cumulants which involve more than three operators he finds a restriction on the form of $C_{1,\ldots,n}(\rho_1,\ldots,\rho_n)$. For two-point cumulants a selection rule has been proposed (Fisher 1973):

$$C_{1,2}(\rho_1,\rho_2)=0 \qquad \text{unless } x_1=x_2. \tag{1.3}$$

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Recently Wolsky and Green (1974) and Wolsky *et al* (1973) have discussed the connection between conformal and dilatation covariance within the framework of classical statistical mechanics.

In this paper we study conformal convariance within the framework of Wilson's renormalization group approach (Wilson and Kogut 1974). Our motivation is twofold. We want to give a fairly complete discussion of the implications of conformal covariance which involve eigen-operators of the linearized renormalization group equation. We are led to extend the analysis to operators which have arbitrary tensor properties under spatial rotations. Even for scalar operators we find corrections to the simple results (1.2) and (1.3). Our second purpose is to formulate the conditions which the fixed point Hamiltonian of the renormalization group must satisfy if conformal covariance is to hold.

The idea of our approach is simple. It has been shown (Wegner 1974) that Wilson's renormalization group operator can be constructed by a transformation of variables in the functional integral which defines the partition function $Z[H]$ of the Hamiltonian H. An important ingredient of this transformation is a dilatation in r space. We construct a conformal operator K along the same lines, substituting the infinitesimal dilatation by an infinitesimal conformal transformation. By construction, the generating functional $\ln Z[H+\Sigma\alpha_i A_i(\rho_i)]$ of the cumulants is invariant with respect to K. This yields identities among cumulants involving $K[H]$ and $K[A_i(\rho)]$. We expand $K[H]$ and $K[A_i(\rho)]$ with respect to eigen-operators of the linearized renormalization group equation, and we evaluate all terms of the identities using the asymptotic behaviour (1.1). In this way we establish relations among the coefficients $C_{1,\ldots,n}(\rho_1,\ldots,\rho_n)$, which can be evaluated. Our treatment is formal, since we do not discuss convergence problems, and we will freely interchange limiting processes.

In § 2 we establish our notation and formulate the model. We give the scaling form of the cumulants in ordinary (r) space, which follows from the renormalization group equation.

In § 3 we construct the conformal operator K, and evaluate $K[\mathcal{O}]$ for short-range operators $\mathcal{O}$. The implications for the cumulants are evaluated in § 4, which contains our main results. Some details are given in the appendix. In § 5 we study surface effects and § 6 contains a summary.

2. Results of the renormalization group approach

2.1. The model

We consider a real classical spin field $S^\alpha(r)$, $\alpha=1,\ldots,n$, defined in d dimensional r space. In the following we omit the spin indices, since they are of no importance for our discussion. The Hamiltonian is written as a functional of $S(r)$:

$$H[S]=\sum_{m=0}^{\infty}\frac{1}{m!}\int \mathrm{d}^d r_1\ldots\mathrm{d}^d r_m \prod_{j=1}^{m} S(r_j)h(r_1,\ldots,r_m). \tag{2.1}$$

The first term ($m=0$) is understood to be a numerical constant. The integration implies summation over spin indices. Since we confine ourselves to the infinite volume limit, we assume translational invariance:

$$h(r_1,\ldots,r_m)=h(0,r_2-r_1,\ldots,r_m-r_1). \tag{2.2}$$

In the following we omit the dimension d at the differentials. Translational invariant localized operators $A(\rho)$ are defined by

$$A(\rho)[S]=\sum_{m=0}^{\infty}\frac{1}{m!}\int \mathrm{d}r_1\ldots \mathrm{d}r_m \prod_{j=1}^{m} S(r_j)a(\rho, r_1, \ldots, r_m) \tag{2.3}$$

where

$$a(\rho, r_1, \ldots, r_m)=a(0, r_1-\rho, \ldots, r_m-\rho). \tag{2.4}$$

The kernels $h(r_1, \ldots, r_m)$ and $a(\rho, r_1, \ldots, r_m)$ are symmetric under permutation of $\{r_1, \ldots, r_m\}$. The partition function $Z[H]$ is defined as a functional integral over the fields $S(r)$:

$$Z[H]=\int_S \mathrm{e}^{-H[S]}. \tag{2.5}$$

2.2. *The renormalization group operator*†

A special form of the generating operator R of the renormalization group for this model has been given by Wilson and Kogut (1974). Wegner (1974) has shown how R can be constructed by a transformation of variables in the functional integral (2.5). Since in § 3 we construct a conformal operator along the same lines, we here quote the results for $R[H]$ without further comments:

$$R[H]=\int \mathrm{d}r\{-\tfrac{1}{2}\,dS(r)-(r\,.\,\nabla_r S(r))+[(b-2\Delta_r)S(r)]\}\frac{\delta H}{\delta S(r)}+\int \mathrm{d}r(b-2\Delta_x)$$
$$\times\left(\frac{\delta^2 H}{\delta S(r)\,\delta S(x)}-\frac{\delta H}{\delta S(r)}\frac{\delta H}{\delta S(x)}\right)_{x=r}. \tag{2.6}$$

Here ∇_x and Δ_x denote gradient and Laplacian in d dimensional space acting on the variable x and b is a numerical parameter. By construction R satisfies the equation

$$0=\frac{\int_S R[H]\,\mathrm{e}^{-H}}{\int_S \mathrm{e}^{-H}}+C. \tag{2.7}$$

The (divergent) constant C is independent of H, and cancels in the expressions for the correlation functions.

For later use we introduce the notation

$$\{A|B\}=\int \mathrm{d}r\frac{\delta A}{\delta S(r)}(b-2\Delta_r)\frac{\delta B}{\delta S(r)}=\{B|A\} \tag{2.8}$$

where A and B are operators of the type (2.1) or (2.3).

The fixed point Hamiltonian H^* is defined as a solution of the renormalization group equation

$$R[H^*]=C_{H^*} \tag{2.9}$$

where we allow for an arbitrary constant C_{H^*} on the right-hand side. We introduce the

† We here essentially transform the results of Wilson and Kogut (1974, § XI) to r space, which is more suitable for our purpose.

scaling fields g_i (Wegner 1972), which are curvilinear coordinates in the space of Hamiltonians. They have the property

$$e^{tR}H\{g\} = H\{g_i\, e^{y_i t}\} + C_{H^*}t. \tag{2.10}$$

The curly bracket indicates that $H\{g\}$ depends on the sequence $\{g_1, g_2, \ldots\}$. The term $C_{H^*}t$ has been added to allow for the choice

$$H\{0\} = H^*. \tag{2.11}$$

Differentiating equation (2.10) first with respect to t at $t=0$, and then with respect to g_j we find

$$R_{\mathrm{L}}^{\{g\}}[\mathcal{O}_j\{g\}] = \Big(y_j + \sum y_k g_k \frac{\partial}{\partial g_k}\Big)\mathcal{O}_j\{g\} \tag{2.12}$$

$$\mathcal{O}_j\{g\} = \frac{\partial}{\partial g_j} H\{g\}. \tag{2.13}$$

We have introduced the linearized renormalization group operator

$$R_{\mathrm{L}}^{\{g\}}[\mathcal{O}] = \int \mathrm{d}r\Big[-\frac{d}{2}S(r) - (r\,.\,\nabla_r S(r)) + [(b-2\Delta_r)S(r)] + \frac{\delta}{\delta S(r)}(b-2\Delta_r) - \Big((b-2\Delta_r)\frac{\delta H\{g\}}{\delta S(r)}\Big) - \frac{\delta H\{g\}}{\delta S(r)}(b-2\Delta_r)\Big]\frac{\delta \mathcal{O}}{\delta S(r)}. \tag{2.14}$$

At the fixed point equation (2.12) yields

$$R_{\mathrm{L}}^*[\mathcal{O}_j^*] = y_j \mathcal{O}_j^* \tag{2.15}$$

where we use an asterisk to denote quantities which are evaluated at $\{g_i = 0\}$. We call $\mathcal{O}_i\{g\}$ a (not localized) eigen-operator of $R_{\mathrm{L}}^{\{g\}}$. Localized eigen-operators $A_j(\rho, \{g\})$ are defined by the equations

$$R_{\mathrm{L}}^{\{g\}}[A_j(\rho, \{g\})] = \Big(x_j + \sum_i y_i g_i \frac{\partial}{\partial g_i} - \rho\nabla_\rho\Big)A_j(\rho, \{g\}) \tag{2.16}$$

$$R_{\mathrm{L}}^*[A_j^*(\rho)] = (x_j - \rho\nabla_\rho)A_j^*(\rho). \tag{2.17}$$

We confine ourselves to fixed points which have the following properties.

(Ri) The kernels of H^* and of all eigen-operators $\mathcal{O}_j^*$, $A_j^*(\rho)$ are of short range, i.e. they vanish at least exponentially if the distance between any two arguments r_i, r_j or r_i, ρ tends to infinity.

(Rii) The kernels of H^* are invariant both with respect to rotations in r space and with respect to a reflection at $r=0$.

(Riii) There exists a real number M with the property

$$M > \mathrm{Re}\, x_i \qquad \text{for all } i \tag{2.18}$$

where Re x denotes the real part of x.

Property (i) will be at the basis of most arguments used in that paper. By virtue of property (ii) the operator R_{L} commutes with rotations and point reflections. We therefore can choose the kernels of $\mathcal{O}_i^*$ and $A_i^*(\rho)$ to have definite tensor properties (parity) with respect to rotations (reflections) centred at $r=0$ or at $r=\rho$, respectively. To begin with we will assume the existence of complete sets of eigen-solutions of equations (2.15) and (2.17). In § 4.3 we generalize our treatment to include the case where R_L^* is of the Jordan normal form:

$$R_L^*[A_{i,n}^*(\rho)] = (x_i - \rho\nabla_\rho)A_{i,n}^*(\rho) + A_{i,n-1}^*(\rho)$$
$$A_{i,n}^*(\rho) = 0 \qquad \text{for } n < 0. \tag{2.19}$$

Corresponding equations hold for $\mathcal{O}_{i,n}^*$. Finally we will argue that our treatment does not rely on the completeness assumptions.

2.3. *Asymptotic form of the cumulants*

We apply the operator R to the Hamiltonian

$$H\{g, \alpha\} = H\{g\} + \alpha \prod_{i=1}^{n} A_i(\rho_i, \{g\}). \tag{2.20}$$

This yields

$$R[H\{g, \alpha\}] = R[H\{g\}] + \alpha \sum_{i=1}^{n} \prod_{j\neq i} A_j(\rho_j, \{g\}) R_L^{\{g\}}[A_i(\rho_i, \{g\})] + 2\alpha \sum_{i<j} \prod_{k\neq i,j} A_k(\rho_k, \{g\})$$
$$\times \{A_i(\rho_i, \{g\}) | A_j(\rho_j, \{g\})\} + O(\alpha^2). \tag{2.21}$$

We substitute these expressions into equation (2.7), and we differentiate with respect to α. Using equations (2.10) and (2.16) we find (compare Wegner 1975, equation (5.20))

$$0 = \left(\sum_{i=1}^{n} (x_i - \rho_i\nabla_i) + \sum_l y_l g_l \frac{\partial}{\partial g_l}\right) \left\langle \prod_{j=1}^{n} A_j(\rho_j, \{g\}) \right\rangle^{\{g\}} + 2 \sum_{i<j} \left\langle \prod_{k\neq i,j} A_k(\rho_k, \{g\}) \right.$$
$$\left. \times \{A_i(\rho_i, \{g\}) | A_j(\rho_j, \{g\})\} \right\rangle^{\{g\}}. \tag{2.22}$$

Here $\langle \ldots \rangle^{\{g\}}$ denotes the expectation value with respect to $H\{g\}$. We transform equation (2.22) into a differential equation in t:

$$0 = \frac{d}{dt}\left[e^{xt} \left\langle \prod_{j=1}^{n} A_j(\rho_j\, e^{-t}, \{g_l\, e^{y_l t}\}) \right\rangle^{\{g_l e^{y_l t}\}} \right] + 2\, e^{xt} \sum_{i<j} \left\langle \prod_{k\neq i,j} A_k(\rho_k\, e^{-t}, \{g_l\, e^{y_l t}\}) \right.$$
$$\left. \times \{A_i(\rho_i\, e^{-t}, \{g_l\, e^{y_l t}\}) | A_j(\rho_j\, e^{-t}, \{g_l\, e^{y_l t}\})\} \right\rangle^{\{g_l e^{y_l t}\}} \tag{2.23}$$

where

$$x = \sum_{i=1}^{n} x_i. \tag{2.24}$$

Integrating and changing some notation we find an identity for the disconnected correlation functions:

$$\left\langle \prod_{j=1}^{n} A_j(\rho_j s, \{g_l(s)\}) \right\rangle^{\{g_l(s)\}}$$
$$= s^{x} \left(\left\langle \prod_{j=1}^{n} A_j(\rho_j, \{g\}) \right\rangle^{\{g\}} + 2 \sum_{i<j} \int_1^s \frac{dz}{z} z^{-x} \left\langle \prod_{k\neq i,j} A_k(\rho_k z, \{g_l(z)\}) \right. \right.$$
$$\left.\left. \times \{A_i(\rho_i z, \{g_l(z)\}) | A_j(\rho_j z, \{g_l(z)\})\} \right\rangle^{\{g_l(z)\}} \right) \tag{2.25}$$

with $s>0$, arbitrary. We have introduced the notation

$$g_l(s)=g_l s^{-y_l}. \tag{2.26}$$

Equation (2.25) implies that $\langle A_i^*(\rho)\rangle^*$ vanishes unless $x_i=0$. (Note that the expectation values are translational invariant.) For operators with Re $x_i<0$ this result is easily extended to the whole critical surface which is defined by the relation

$$g_l=0 \qquad \text{if } y_l>0. \tag{2.27}$$

For the two- and three-point cumulants equation (2.25) yields identities which formally may be found by substituting all expectation values by the cumulant average $\langle\ldots\rangle_c$.

At $\{g_i=0\}$ the integrands which occur in equation (2.25) are assumed to be of short range in z, provided that all points ρ_i are distinct. To justify this assumption we note that the operator $\{A_i^*(\rho_i z)|A_j^*(\rho_j z)\}$ which occurs in the integrands, has kernels which decrease exponentially with increasing $|\rho_i-\rho_j|z$. As a function of s the integrals therefore are taken to equal a constant (which is equal to the value at $s=\infty$) plus a short-range term (SHR(s)). In order to extend our analysis to a neighbourhood of the fixed point we furthermore assume that the additional z dependence due to $\{g_l(z)\}$ does not spoil the exponential decrease of the integrands. As a consequence the cumulants show scaling behaviour (Fisher 1973, equation (52)):

$$\left\langle\prod_{i=1}^{n} A_i(\rho_i s,\{g_l(s)\})\right\rangle_c^{\{g_l(s)\}}=s^{x}C_{1,\ldots,n}(\rho_1,\ldots,\rho_n,\{g\})+\mathrm{SHR}(s) \qquad n=2,3 \tag{2.28}$$

For physical reasons it is obvious that the cumulants at H^* should vanish in the limit $s\to\infty$. This justifies assumption (Riii), and it implies that we can choose $M=0$ in equation (2.18) if we allow for one exception: in view of equations (2.14) and (2.17), a numerical constant can be taken to be an eigen-operator with $x_0=0$.

We also need the asymptotic form at H^* of the cumulants of a product of one non-localized operator with two or three localized operators. From equations (2.28) and (2.13) we find by differentiation with respect to g_k

$$\left\langle\prod_{i=1}^{n} A_i^*(\rho_i s)\mathcal{O}_k^*\right\rangle_c^* = C^*_{1,\ldots,n;k}(\rho_1,\ldots,\rho_n)s^{y_k+x}+\sum_{j=1}^{n}\left\langle\prod_{\substack{i=1\\ i\neq j}}^{n} A_i^*(\rho_i s)\frac{\mathrm{d}}{\mathrm{d}g_k}A_j(\rho_j s,\{g\})\big|_{\{0\}}\right\rangle_c^*$$
$$+\mathrm{SHR}(s) \tag{2.29}$$

We expand $\mathrm{d}A_j(\rho,\{g\})|_{\{0\}}/\mathrm{d}g_k$ according to

$$\frac{\mathrm{d}}{\mathrm{d}g_k}A_j(\rho,\{0\})=\sum_l \gamma_{jk}^l A_l^*(\rho) \tag{2.30}$$

and we use equation (2.28) at H^*. This yields

$$\left\langle\prod_{i=1}^{n} A_i^*(\rho_i)\mathcal{O}_k^*\right\rangle_c^*$$
$$=C^*_{1,\ldots,n;k}(\rho_1,\ldots,\rho_n)s^{y_k+x}$$
$$+\sum_l\sum_{j=1}^{n}\gamma_{jk}^l C^*_{1,\ldots,j-1,l,j+1,n}(\rho_1,\ldots,\rho_n)s^{x_l+x-x_j}+\mathrm{SHR}(s). \tag{2.31}$$

We here have assumed that equation (2.28) can be differentiated at $\{0\}$. Equation

(2.16) allows us to calculate $(d/dg_k)A_j$ in terms of $\{A_j^*(\rho)|\mathcal{O}_k^*\}$. We find that γ_{jk}^l diverges logarithmically if the relation

$$x_l - x_j - y_k = 0 \tag{2.32}$$

holds. It is possible, but lengthy, to determine the asymptotic behaviour of $(\Pi A_i^*(\rho_i)\mathcal{O}_u^*)_c^*$ directly at the fixed point. If equation (2.32) holds we find logarithmic corrections in s to the right-hand side of equation (2.31). In the following we will neglect this complication.

3. The conformal operator

3.1. Construction

Our construction of the conformal operator closely follows the construction of R given by Wegner (1974). We define a vector-valued function $u_l(r, \eta)$, such that the mapping

$$r \to r + \eta u_l(r, \eta) \tag{3.1}$$

is one-to-one for small values of η. For $\eta > 0$ $u_l(r, \eta)$ is a smooth function of r, which vanishes identically for $|r| \geq l$. The mapping (3.1) induces a change of $S(r)$:

$$S(r) \to S(r) + \eta u_l(r, \eta)\nabla_r S(r) + O(\eta^2). \tag{3.2}$$

We in addition add a function $\psi(r, S)$ which depends functionally on S, and is specified below:

$$S'(r) = S(r) + \eta u_l(r, \eta)\nabla_r S(r) + \eta\psi(r, S). \tag{3.3}$$

We substitute $S'(r)$ for $S(r)$ in $Z[H]$. Obviously H changes according to

$$H[S] = H[S'] - \eta \int dr(u_l(r, \eta)\nabla_r S'(r) + \psi(r, S'))\frac{\delta H}{\delta S'(r)} + O(\eta^2). \tag{3.4}$$

The transformation (3.3) changes the measure of the functional integration. To determine this contribution we confine the system to a finite volume, and we express $Z[H]$ as a multiple integral over the discrete set of Fourier components S_q of $S(r)$. The change in the integration measure $\Pi\, dS_q$ is given by the functional determinant of the (Fourier-transformed) transformation (3.3). Using the fact that $S(r)$ is real and that $dS_{-q}\, dS_q$ is to be interpreted as $d\,\mathrm{Re}\, S_q\, d\,\mathrm{Im}\, S_q$, we find (compare Wegner 1974, equation (2.6))

$$\Pi\, dS_q = \Pi\, dS_q'\left[\exp\left(-\eta\sum_q \frac{\partial\psi_q[S']}{\partial S_q'}\right) + O(\eta^2)\right]. \tag{3.5}$$

Combining equations (3.4) and (3.5) we find in r space

$$Z[H] = Z[H'] + O(\eta^2) \tag{3.6}$$

$$H'[S] = H[S] - \eta \int dr\left((u_l(r, \eta)\nabla_r S(r) + \psi(r, S))\frac{\delta H}{\delta S(r)} + \frac{\delta\psi(r, S)}{\delta S(r)}\right). \tag{3.7}$$

We want to use the defining equations of H^* and $A_i^*(\rho)$ to evaluate the result of the conformal transformation. We therefore choose $\psi(r, S)$ in such a way that for†

† This choice, strictly speaking, is forbidden by $u_l(r, \eta) = 0$ for $|r| \geq l$.

$u_l(r,\eta)=r$ we recover the renormalization group operator. The following ansatz proves to be adequate:

$$\psi(r,S)=\lambda_l(r,\eta)\left(\frac{d}{2}S(r)+(b-2\Delta_r)\frac{\delta H}{\delta S(r)}\right)-(b-2\Delta_r)(\lambda_l(r,\eta)S(r)). \tag{3.8}$$

The function $\lambda_l(r,\eta)$ is specified below. We substitute the resulting operator H' into equation (3.6) and we differentiate with respect to η at $\eta=0$. This yields

$$0=\frac{\int_s K^l[H]\,\mathrm{e}^{-H}}{\int_s \mathrm{e}^{-H}}+C_K \tag{3.9}$$

$$K^l[H]=\int \mathrm{d}r\left[\left(-\frac{d}{2}\lambda_l(r,0)S(r)-u_l(r,0)(\nabla_r S(r))+[(b-2\Delta_r)(\lambda_l(r,0)S(r))]\right.\right.$$
$$\left.\left.+\lambda_l(r,0)\frac{\delta}{\delta S(r)}(b-2\Delta_r)\right)\frac{\delta H}{\delta S(r)}-\frac{\delta H}{\delta S(r)}\lambda_l(r,0)(b-2\Delta_r)\frac{\delta H}{\delta S(r)}\right]. \tag{3.10}$$

The constant C_K is infinite. This divergence has no influence on the cumulants, and we can avoid its occurrence at intermediate steps by using a regularized form of $\psi(r,S)$.

The infinitesimal conformal transformation is defined by

$$u(r)=ar^2-2(ar)r \tag{3.11}$$

where a is an arbitrary vector. We define the operator K^l by the choice

$$u_l(r,\eta)=u(r)H^l_\eta(r) \tag{3.12}$$

$$\lambda_l(r,\eta)=\lambda(r)H^l_\eta(r) \tag{3.13}$$

$$\lambda(r)=\frac{1}{d}\operatorname{div}u(r)=-2(ar) \tag{3.14}$$

$$\lim_{\eta\to 0}H^l_\eta(r)=\theta(l-|r|)=\begin{cases}1 & \text{for } |r|<l\\ 0 & \text{otherwise.}\end{cases} \tag{3.15}$$

For $\eta=0$ this operator describes a conformal mapping in the region $|r|<l$. In § 4 we discuss the bulk effects by evaluating the operator K which is found from equation (3.10) by the substitution

$$u_l(r,0)\to u(r),\qquad \lambda_l(r,0)\to\lambda(r). \tag{3.16}$$

The difference K^l-K (surface effects) is treated in § 5.

3.2. Evaluation of K[H]

To evaluate the consequences of equation (3.9) at the fixed point we apply K to the Hamiltonian $H^*+\Sigma\alpha^iA_i^*(\rho_i)$. We decompose $K[H]$ according to

$$K[H^*+\textstyle\sum\alpha^iA_i^*(\rho_i)]=K[H^*]+\sum\alpha^iK_L[A_i^*(\rho_i)]-\sum\alpha^i\alpha^jK_Q[A_i^*(\rho_i),A_j^*(\rho_j)]. \tag{3.17}$$

The structure of K_L is completely analogous to that of R_L (equation (2.14)). The term K_Q is of short range in $|\rho_i-\rho_j|$. We evaluate $K_L[A_i^*(\rho_i)]$ by subtracting equation (2.17) multiplied by $\lambda(\rho)$. Similarly we subtract from $K[H^*]$ an equation based on equation (2.9). The details may be found in the appendix (§ A.1). We use the invariance (Rii) to derive the following results.

$$K[H^*]=\sum_{m=3}^{\infty}\frac{1}{m!}\int \mathrm{d}r_1\ldots\mathrm{d}r_m\prod_{j=1}^{m}S(r_j)\Bigg\{\sum_{i=1}^{m}[u(r_i-R)\,.\,\nabla_i-2\lambda(r_i-R)\Delta_i]$$
$$\times h^*(r_1,\ldots,r_m)+\int \mathrm{d}r\,\lambda(r-R)(b-2\Delta_x)\Bigg[h^*(x,r,r_1,\ldots,r_m)-\sum_{n=0}^{m}\binom{m}{n}$$
$$\times h^*(r,r_1,\ldots,r_n)h^*(x,r_{n+1},\ldots,r_m)\Bigg]_{x=r}\Bigg\}+C_K[H^*] \tag{3.18}$$
$$R=\frac{1}{m}\sum_{k=1}^{m}r_k.$$

The constant $C_K[H^*]$ cancels in the evaluation of cumulants:

$$K_L[A_i^{*\{\alpha\}}(\rho)]=(\lambda(\rho)x_i-u(\rho)\,.\,\nabla_\rho)A_i^{*\{\alpha\}}(\rho)+2\frac{\partial}{\partial\epsilon_-}\Omega(A_i^{*\{\alpha\}}(\rho))|_{\epsilon=0}+\delta K_L(A_i^{*\{\alpha\}}(\rho)) \tag{3.19}$$

$$\Omega(A_i^{*\{\alpha\}}(\rho))=\sum_{\beta_1\ldots\beta_t=1}^{d}\prod_{\nu=1}^{t}\Omega_{\alpha_\nu\beta_\nu}(\epsilon,a,\rho)A^{*\{\beta_1,\ldots,\beta_t\}}(\rho) \tag{3.20}$$

$$\Omega_{\alpha\beta}(\epsilon,a,\rho)=\delta_{\alpha\beta}+\epsilon(a_\alpha\rho_\beta-\rho_\alpha a_\beta) \tag{3.21}$$

$$\delta K_L[A_i^*(\rho)]=\int \mathrm{d}r\Bigg\{-u(r-\rho)\,.\,(\nabla_r S(r))+(b-2\Delta_r)(\lambda(r-\rho)S(r))$$
$$+\lambda(r-\rho)\Bigg[-\frac{d}{2}S(r)+\frac{\delta}{\delta S(r)}(b-2\Delta_r)$$
$$-\left((b-2\Delta_r)\frac{\delta H^*}{\delta S(r)}\right)-\frac{\delta H^*}{\delta S(r)}(b-2\Delta_r)\Bigg]\Bigg\}\frac{\delta A_i^*(\rho)}{\delta S(r)}. \tag{3.22}$$

The superscript $\{\alpha\}=\{\alpha_1,\ldots,\alpha_t\}$ represent the tensor indices of $A_i^*(\rho)$. The contribution $\delta K[A_i^*(\rho)]$ is a localized operator of type (2.3).

For the subsequent discussion it is important that $K[H^*]$ and $K_L[A_i^*(\rho)]$ are completely reduced to translational invariant operators of the type (2.1) or (2.3). We furthermore note that $K[H^*]$ is a vector-type operator whereas the tensor-rank of $\delta K_L[A_i^*(\rho)]$ exceeds that of $A_i^*(\rho)$ by one. To show this we refer to equations (3.11) and (3.14), and we note that the vector a is a fixed parameter. Introducing the vector components a_δ of a we write

$$K[H^*]=\sum_{\delta=1}^{d}a_\delta K^\delta[H^*] \tag{3.23}$$

$$K_L[A_i^{*\{\alpha\}}(\rho)]=\sum_{\delta=1}^{d}a_\delta\delta K_L^\delta[A_i^{*\{\alpha\}}(\rho)]. \tag{3.24}$$

4. Consequences of conformal covariance

Differentiating equation (3.9) with respect to a set of parameters α^i of the Hamiltonian $H[\alpha]$ we establish identities which express the response of the cumulants to an infinitesimal conformal transformation. Since a complete characterization of $\langle A_i^*(\rho)\rangle^*$ is provided by the renormalization group itself (compare the discussion in connection with equation (2.25)), we concentrate on the two- and three-point cumulants. We work exclusively at the fixed point, and we therefore omit the superscript.

4.1. Two-point cumulants

Equations (3.9), (3.17) and (3.19) yield

$$\sum_{i=1}^{2} [2(a\rho_i)x_i + (a\rho_i^2 - 2(a\rho_i)\rho_i)\nabla_i]\langle A_1(\rho_1)A_2(\rho_2)\rangle_c - 2(\partial/\partial\epsilon)\langle\Omega[A_1(\rho_1)]\Omega[A_2(\rho_2)]\rangle_c|_{\epsilon=0}$$
$$= -\langle A_1(\rho_1)A_2(\rho_2)K[H^*]\rangle_c + \langle\delta K_L[A_1(\rho_1)]A_2(\rho_2) + A_1(\rho_1)\delta K_L[A_2(\rho_2)]\rangle_c$$
$$+ \langle K_Q[A_1(\rho_1), A_2(\rho_2)] + K_Q[A_2(\rho_2), A_1(\rho_1)]\rangle. \tag{4.1}$$

To get a better feeling for the structure of this equation, we exhibit the mechanism which guarantees the translational invariance. Substituting ρ_i by $\rho_i'+b$ we find the following terms, ordered according to powers of b:

(i) b^2 $\qquad (ab^2 - 2(ab)b)(\nabla_1 + \nabla_2)\langle A_1(\rho_1')A_2(\rho_2')\rangle_c.$ (4.2)

This vanishes by virtue of the translational invariance of the cumulants.

(ii) b^1
$$2(ab)[(x_1 + x_2 - \rho_1'\nabla_1 - \rho_2'\nabla_2)\langle A_1(\rho_1')A_2(\rho_2')\rangle_c + 2\langle\{A_1(\rho_1')|A_2(\rho_2')\}\rangle]$$
$$+2\frac{\partial}{\partial\epsilon}\Big(\langle A_1^{\{\alpha\}}(\Omega(\epsilon, a, b)\rho_1')A_2^{\{\beta\}}(\Omega(\epsilon, a, b)\rho_2')\rangle_c - \sum_{\{\alpha',\beta'\}}\prod_\nu \Omega_{\alpha_\nu\alpha_\nu'}(\epsilon, a, b)$$
$$\times\prod_\mu \Omega_{\beta_\mu\beta_\mu'}(\epsilon, a, b)\langle A_1^{\{\alpha'\}}(\rho_1')A_2^{\{\beta'\}}(\rho_2')\rangle_c. \tag{4.3}$$

The contribution proportional to (ab) vanishes by virtue of the renormalization group equation (compare equation (2.22)). The second part vanishes in view of the tensor properties of $A_i^{\{\alpha\}}(\rho)$.

(iii) b^0. These terms yield equation (4.1), written for ρ_i'. In view of the translational invariance we simplify equation (4.1) by choosing $\rho_1 = 0$, $\rho_2 = re^0$, where e^0 denotes a unit vector. Assuming completeness of the sets of eigen-operators we expand

$$K^\delta[H^*] = \sum_k b_*^k \mathcal{O}_k^{(\delta)} \tag{4.4}$$

$$\delta K^\delta[A_i^{\{\alpha\}}(\rho)] = \sum_k \gamma_i^k A_k^{\{\delta,\alpha\}}(\rho). \tag{4.5}$$

We combine equations (4.1), (4.4) and (4.5) with the asymptotic form (2.28), (2.31) of the cumulants. This yields the basic identity:

$$[2r(ae^0)x_2 + r^2(a - 2(ae^0)e^0)\,.\,\nabla_{re^0}]C_{1,2}^{\{\alpha,\beta\}}(e^0)r^{x_1+x_2}$$
$$-2\frac{\partial}{\partial\epsilon}\sum_{\{\beta'\}}\prod_\nu \Omega_{\beta_\nu\beta_\nu'}(\epsilon, a, e^0)C_{1,2}^{\{\alpha,\beta'\}}(e^0)r^{x_1+x_2+1}$$
$$= -\sum_k\sum_\delta b_*^k a_\delta C_{1,2;k}^{\{\alpha,\beta,\delta\}}r^{y_k+x_1+x_2} + \sum_k\sum_\delta a_\delta\{b_1^k C_{k,2}^{\{\delta\alpha,\beta\}}(e^0)r^{x_k+x_2}$$
$$+ b_2^k C_{1,k}^{\{\alpha,\delta\beta\}}(e^0)r^{x_1+x_k}\} \tag{4.6}$$

$$b_i^k = \gamma_i^k + \sum_l b_*^l \gamma_{il}^k. \tag{4.7}$$

We have omitted all short-range terms, including the expectation value of $T_q[A_1(\rho_1), A_2(\rho_2)]$, and we have indicated at the coefficients C the tensor indices of the operators involved ($\{\delta\alpha, \beta\} = \{\delta, \alpha_1, \ldots, \alpha_{t_1}, \beta_1, \ldots, \beta_{t_2}\}$, for instance).

Equation (4.6) is simplified if we choose $a = e^0$. A simple calculation yields

$$\begin{aligned}(x_2 - x_1) C_{1,2}^{\{\alpha,\beta\}}(e^0) = &-\sum_k \sum_\delta b_*^k e_\delta^0 C_{1,2;k}^{\{\alpha,\beta,\delta\}}(e^0) r^{y_k - 1} \\ &+ \sum_k \sum_\delta e_\delta^0 [b_1^k C_{k,2}^{\{\delta\alpha,\beta\}}(e^0) r^{x_k - x_1 - 1} + b_2^k C_{1,k}^{\{\alpha,\delta\beta\}}(e^0) r^{x_k - x_2 - 1}].\end{aligned} \tag{4.8}$$

We distinguish two possibilities.

(i) There exists an operator $\mathcal{O}_k^{(\delta)}$ with $y_k = +1$, $b_*^k \neq 0$. This is a property of H^*. Equation (4.8) at best establishes a connection between $C_{1,2}(e^0)$ and $C_{1,2;k}(e^0)$.

(ii) There exists no such operator. This is the case of interest.

(iia) $(x_2 - x_1)$ is not an integer.

Assuming that $C_{1,2}(e^0)$ does not vanish identically, we conclude that the second sum on the right-hand side of equation (4.8) contains a non-vanishing term independent of r. Therefore there exists an operator $A_k(\rho)$ with the properties

$$\left.\begin{aligned} &x_k = x_i + 1 \\ &C_{\bar{\imath},k}(e^0) \neq 0 \end{aligned}\right\} i = 1, \bar{\imath} = 2 \qquad \text{or } i = 2, \bar{\imath} = 1. \tag{4.9}$$

According to our assumption $x_k - x_{\bar{\imath}}$ does not vanish, and we can apply the same argument to $C_{\bar{\imath}k}(e^0)$. We conclude that there exist non-vanishing eigen-operators with arbitrary large positive eigenvalues, which contradicts assumption (Riii). Thus $C_{1,2}(e^0)$ vanishes identically.

(iib) $x_2 - x_1 = m$, m integer.

The construction used above, which was based on equation (4.8), terminates whenever x_1 and x_2 become equal. We therefore discuss the conditions under which the left-hand side of the full equation (4.6) vanishes for $x_1 = x_2$. The component of a parallel to e^0 cancels, and we are left with the equation

$$re^\perp \nabla_{re^0} C_{1,2}^{\{\alpha,\beta\}}(e^0) - 2\frac{\partial}{\partial\epsilon} \sum_{\{\beta'\}} \prod_\nu \Omega_{\beta_\nu\beta'_\nu}(\epsilon, e^\perp, e^0) C_{1,2}^{\{\alpha,\beta'\}}(e^0) = 0. \tag{4.10}$$

Here $e^\perp$ denotes a unit vector orthogonal to e^0. We show in the appendix (§ A.2) that a non-vanishing solution of this equation has positive parity:

$$C_{1,2}^{\{\alpha,\beta\}}(-e^0) = +C_{1,2}^{\{\alpha,\beta\}}(+e^0). \tag{4.11}$$

On the other hand the parity of $C_{1,2}^{\{\alpha,\beta\}}(e^0)$ is determined by the operators $A_1^{\{\alpha_1,\ldots,\alpha_{t_1}\}}(0)$, $A_2^{\{\beta_1,\ldots,\beta_{t_2}\}}(re^0)$ according to

$$C_{1,2}^{\{\alpha,\beta\}}(-e^0) = (-1)^{t_1 + t_2 + \pi_1 + \pi_2} C_{1,2}^{\{\alpha,\beta\}}(e^0) \tag{4.12}$$

$$\pi_i = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \text{ if } A_i^{\{\alpha\}}(\rho) \text{ is a } \begin{Bmatrix} \text{tensor} \\ \text{pseudotensor} \end{Bmatrix}. \tag{4.13}$$

We define an index z_i by

$$z_i = x_i + t_i + \pi_i. \tag{4.14}$$

For a non-vanishing cumulant, at which the construction of §(ii$a$) terminates, we find, according to equations (4.11) and (4.12),

$$z_1 - z_2 = t_1 + \pi_1 - t_2 - \pi_2 = t_1 + t_2 + \pi_1 + \pi_2 - 2(t_2 + \pi_2) = 2m, \qquad m \text{ integer.} \tag{4.15}$$

Since equation (4.6) couples only operators whose indices z_i differ by $2m$, m integer, we have proved the following selection rule:

$$C_{1,2}^{\{\alpha,\beta\}}(e^0) = 0 \qquad \text{unless } z_1 - z_2 = 2m, \qquad m \text{ integer.} \tag{4.16}$$

Equation (4.16) generalizes and corrects the result (1.3). It incorporates just the additional features which are necessary to avoid a contradiction. From equation (2.17) it is obvious that $\Delta_\rho A_i(\rho)$ is a scalar eigen-operator with the eigenvalue $x_i - 2$, provided $A_i(\rho)$ is a scalar eigen-operator with the eigenvalue x_i. Applying equation (1.3) we find

$$0 \sim \langle A_i(0)\Delta_\rho A_i(\rho)\rangle_c = \Delta_\rho \langle A_i(0)A_i(\rho)\rangle_c \sim C_{i,i}(e^0) r^{2x_i - 2} \tag{4.17}$$

which is wrong in general. Equation (4.16) corrects for this contradiction.

4.2. *Three-point cumulants*

For three-point cumulants we get the following identity:

$$\begin{aligned}
&\sum_{i=1}^{3} [2(a\rho_i)x_i + (a\rho_i^2 - 2(a\rho_i)\rho_i)\nabla_i] C_{1,2,3}^{\{\alpha,\beta,\gamma\}}(\rho_1, \rho_2, \rho_3) - 2\frac{\partial}{\partial\epsilon} \sum_{\{\alpha',\beta',\gamma'\}} \prod_{\nu} \Omega_{\alpha_\nu\alpha'_\nu}(\epsilon, a, \rho_1) \\
&\qquad \times \prod_{\mu} \Omega_{\beta_\mu\beta'_\mu}(\epsilon, a, \rho_2) \prod_{\sigma} \Omega_{\gamma_\sigma\gamma'_\sigma}(\epsilon, a, \rho_3) C_{1,2,3}^{\{\alpha',\beta',\gamma'\}}(\rho_1, \rho_2, \rho_3) \\
&\qquad = -\sum_k \sum_\delta b_*^k a_\delta C_{1,2,3;k}^{\{\alpha,\beta,\gamma,\delta\}}(\rho_1, \rho_2, \rho_3) S^{y_k - 1} + \sum_k \sum_\delta a_\delta \{ b_1^k C_{k,2,3}^{\{\delta\alpha,\beta,\gamma\}}(\rho_1, \rho_2, \rho_3) S^{x_k - x_1 - 1} \\
&\qquad + b_2^k C_{1,k,3}^{\{\alpha,\delta\beta,\gamma\}}(\rho_1, \rho_2, \rho_3) S^{x_k - x_2 - 1} \\
&\qquad + b_3^k C_{1,2,k}^{\{\alpha,\beta,\delta\gamma\}}(\rho_1, \rho_2, \rho_3) S^{x_k - x_3 - 1} \} \qquad S \text{ large.}
\end{aligned} \tag{4.18}$$

Straightforward differentiation of equation (3.9) yields a result corresponding to equation (4.1). To derive equation (4.18) we substitute ρ_i by $\rho_i \,.\, S$ and go to the limit of large S.

As in § 4.1 (i) the existence of an operator $\mathcal{O}_k$ with $y_k = +1$, $b_*^k \neq 0$, renders equation (4.18) useless. We thus assume (ii) that the first sum on the right-hand side of equation (4.18) does not contribute a term which is independent of S. By virtue of assumption (Riii) we conclude that there exist sets of operators, for which the left-hand side of equation (4.18) vanishes

$$\begin{aligned}
&\sum_{i=1}^{3} [2(a\rho_i)x_i + (a\rho_i^2 - 2(a\rho_i)\rho_i) \,.\, \nabla_i] C_{1,2,3}^{\{\alpha,\beta,\gamma\}}(\rho_1, \rho_2, \rho_3) \\
&\qquad - \frac{\partial}{\partial\epsilon}\left(\sum \prod \Omega\right) C_{1,2,3}^{\{\alpha',\beta',\gamma'\}}(\rho_1, \rho_2, \rho_3) = 0.
\end{aligned} \tag{4.19}$$

If all three operators are scalar, the second term of this equation vanishes, and we find the well known expression (1.2)

$$C_{1,2,3}(\rho_1, \rho_2, \rho_3) = \Gamma_{1,2,3} |\rho_{12}|^{\Delta_{12,3}} |\rho_{13}|^{\Delta_{13,2}} |\rho_{23}|^{\Delta_{23,1}} \tag{4.20}$$

$$|\rho_{ij}|=|\rho_i-\rho_j|; \qquad \Delta_{ij,k}=x_i+x_j-x_k. \tag{4.21}$$

Equation (4.19) covers the general case. Some examples are given in the appendix (§A.3).

Equation (4.18) again allows for a coupling of an operator $A_i(\rho)$ to an operator $A_k(\rho)$ with $x_k=x_i+1$. From this equation we can evaluate the asymptotic form of the cumulants, provided we know the spectrum of R_L and the tensor properties of the eigen-operators. To give an example we have calculated $C_{1,2,3}(\rho_1,\rho_2,\rho_3)$ under the assumptions that the $A_i(\rho_i)$, $i=1,2,3$, are scalar operators, and that only $A_1(\rho)$ couples to a vector operator $A_4(\rho)$, which in turn is coupled to a scalar $A_5(\rho)$. We find

$$C_{123}(\rho_1,\rho_2,\rho_3)=|\rho_{12}|^{\Delta_{12,3}}|\rho_{13}|^{\Delta_{13,2}}|\rho_{23}|^{\Delta_{23,1}} \times\left(\Gamma_{1,2,3}+b_1^4\Gamma_{4,2,3}\frac{\rho_{31}^2-\rho_{21}^2}{2\rho_{23}^2}+b_1^4b_4^5\Gamma_{5,2,3}\frac{\rho_{31}^2+\rho_{21}^2}{4\rho_{23}^2}\right). \tag{4.22}$$

Without coupling only the first term survives. Again the coupling terms are necessary to cover the case of derivatives with respect to ρ_i. From our examples we expect that the general form of the three-point cumulant is

$$C_{123}^{\{\alpha,\beta,\gamma\}}(\rho_1,\rho_2,\rho_3)=|\rho_{12}|^{\Delta_{12,3}}|\rho_{13}|^{\Delta_{13,2}}|\rho_{23}|^{\Delta_{23,1}}P_{1,2,3}^{\{\alpha,\beta,\gamma\}}\left(\frac{\rho_{12}}{|\rho_{12}|},\frac{\rho_{13}}{|\rho_{13}|},\frac{|\rho_{12}|}{|\rho_{13}|}\right) \tag{4.23}$$

where $P(\ldots)$ denotes a ratio of two finite polynomials which incorporates the tensor properties and the coupling structure.

4.3. *Jordan normal form*

We have repeated the argument of the previous sections under the weaker assumption that R_L can be reduced completely to the form (2.19). The new feature is the occurrence of logarithmic corrections to equation (2.26). The two-point cumulant at H^*, for instance, has the structure

$$\langle A_{x_1,n_1}(0)A_{x_2,n_2}(re^0)\rangle_c=r^{x_1+x_2}\sum_{p=0}^{n_1+n_2}(\ln r)^p \times\sum_{\substack{p_1+p_2=p\\ 0\leqslant p_i\leqslant n_i}}\frac{1}{p_1!p_2!}C_{x_1,n_1-p_1,x_2,n_2-p_2}(e^0)+\mathrm{SHR}(r). \tag{4.24}$$

The selection rule (4.16) remains unchanged. We have found no general restriction for the dependence of $C_{x_1,p_1,x_2,p_2}(e^0)$ on p_1, p_2. For any concrete case, however, restrictions can be worked out by substituting equation (4.24) into the identity (4.1). Since no new general results have emerged from the discussion of the two-point cumulants, we have omitted a discussion of the three-point cumulants, where one expects logarithmic corrections to the behaviour (4.23).

5. Surface effects

In §4 we have neglected the difference between K^l and K. The discussion, given there, remains valid, provided that in the limit of large l (K^l-K) does not create contributions to equations (4.6) or (4.18) which behave like $r^{x_1+x_2+1}$ or S^0, respectively. We here discuss this problem in some more detail.

A typical term of $K^l_{\rm L}[A_i(\rho)]-K_{\rm L}[A_i(\rho)]$ reads

$$-\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{i=1}^{m}\int \mathrm{d}r_1\ldots \mathrm{d}r_m \prod_j S(r_j)\theta(|r_i|-l)u(r_i)\,.\,\nabla_i a_i(\rho, r_1,\ldots, r_m). \tag{5.1}$$

The θ function confines the coordinate r_i to $|r_i|\geqslant l$, and ρ is fixed independent of l. In view of the short range of $a(\rho, r_1, \ldots, r_m)$ we conclude that the contribution (5.1) vanishes for $l\to\infty$. The same argument holds for the other terms of $K^l[A_i(\rho)]-K[A_i(\rho)]$ as well as for $K^l_{\rm Q}-K_{\rm Q}$.

Surface effects can arise from $K^l[H^*]-K[H^*]$, since H^* is not localized and may follow the surface $|r_i|=l$. We demonstrate this for a simple, but typical term

$$\Delta=l^2\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{i=1}^{m}\int \mathrm{d}r_1\ldots \mathrm{d}r_m\prod_{j=1}^{m}S(r_j)\Big(a\,.\frac{r_i}{|r_i|}\Big)\delta(l-|r_i|)h^*(r_1,\ldots,r_m). \tag{5.2}$$

This term arises from the partial integration of†

$$-\int \mathrm{d}r[\theta(|r|-l)u(r)-u(r)]\,.\,(\nabla_r S(r))\frac{\delta}{\delta S(r)}H^* \tag{5.3}$$

(compare equation (A.4)). We define the localized operator $H^*(\rho)$ by the kernels

$$h^*(\rho, r_1,\ldots,r_m)=\frac{1}{m}\sum_{i=1}^{m}\delta(r_i-\rho)h^*(r_1,\ldots,r_m) \tag{5.4}$$

and we expand $H^*(\rho)$ according to

$$H^*(\rho)=\sum c_i A_i(\rho). \tag{5.5}$$

Equation (5.2) transforms into

$$\Delta=l^{d+1}\sum c_i\int \mathrm{d}\Omega_\rho(ae_\rho)A_i(le_\rho). \tag{5.6}$$

Here $\mathrm{d}\Omega_\rho$ denotes the integration over the direction of the unit vector e_ρ in d dimensional space. This terms yields the following contribution to the identity for the two-point cumulants:

$$\begin{aligned}\langle A_1(0)A_2(re^0)\Delta\rangle_c &= l^{d+1}\sum_i c_i\int \mathrm{d}\Omega_\rho(ae_\rho)\langle A_1(0)A_2(re^0)A_i(le_\rho)\rangle_c\\ &=\sum_i c_i r^{x_1+x_2+x_i+d+1}\lambda^{d+1}\int \mathrm{d}\Omega_\rho(ae_\rho)[C_{1,2,i}(0,e^0,\lambda e_\rho)+\ldots]\end{aligned} \tag{5.7}$$

$$\lambda=r^{-1}l. \tag{5.8}$$

We have used equation (2.28). The terms omitted in equation (5.7) are of short range either in r or in λ, as is shown by equation (2.24).

The discussion of § 4.1 becomes valid if in the limit $\lambda\to\infty$ there survives a term of equation (5.7) with $x_i=-d$. Local operators with this eigenvalue in general will exist (compare Wegner 1972, § VII, and references given therein). We have found no satisfactory argument which excludes a contribution of equation (5.7) in the limit $\lambda\to\infty$.

† The term (5.2) is cancelled if we subtract a term $a\,l^2H^l_\eta(r)$ in the definition (3.12). Other terms involving l^2 remain, however, and we therefore have decided to illustrate the surface effects by this simple contribution.

We can only offer some consistency considerations. We use the expressions for $C_{1,2,i}$, $x_i=-d$, derived from conformal covariance to determine the λ dependence in equation (5.7). Assuming that $A_1(\rho)$, $A_2(\rho)$ and $A_i(\rho)$ are scalar, and that none of these three operators couples to an operator $A_k(\rho)$ with $x_k=x_i+1$, $i=1,2$ or $x_k=-d+1$, we find from equation (4.20)

$$C_{1,2,i}(0, e^0, \lambda e_\rho) = \text{constant} \times \lambda^{-2d} + O(\lambda^{-2d-1}). \tag{5.9}$$

This yields

$$\lambda^{d+1} \int d\Omega_\rho(ae_\rho) C_{1,2,i}(0, e^0, \lambda e_\rho)$$
$$= \text{constant} \times \lambda^{-d+1} \int d\Omega_\rho(ae_\rho) + O(\lambda^{-d}) = O(\lambda^{-d}) \underset{\lambda\to\infty}{\to} 0. \tag{5.10}$$

The same result holds in all cases for which we have calculated $C_{1,2,i}$. If $A_i(\rho)$ is coupled to an operator $A_k(\rho)$ we have to use assumption (Riii) with $M=0$: x_k should be strictly negative.

A somewhat lengthy discussion of the other terms of $K'[H^*]-K[H^*]$ yields similar results.

6. Summary and conclusions

We have evaluated the consequences of conformal covariance for a fixed point with the following properties.

(Ri) All operators which occur, are of short range.

(Rii) H^* is invariant with respect to rotations in r space and with respect to reflections at $r=0$.

(Riii) The spectrum of R_L is bounded from above.

With assumptions (Ri) and (Rii) the identities (4.6) and (4.18) hold, possibly corrected by a contribution of the surface effects. Besides (Riii), two additional conditions have to be fulfilled if we want to draw useful conclusions from these identities.

(Ki) The conformal operator K applied to H^* does not create a contribution with the eigenvalue $y=+1$.

(Kii) If the radius of the conformally distorted sphere tends to infinity, any dangerous contribution of the surface effects vanishes.

For such a conformally covariant fixed point we have evaluated the identities assuming that the linearized renormalization group operator has a complete set of eigen-operators. A generalization to the Jordan normal form proved to be possible. Here we want to point out that we do not use the full power of that assumption, but that weaker conditions are sufficient.

(Kiii) There exists a number $N>0$ such that the part of the spectrum of R_L in the half-plane $\text{Re}\, x > -N$ consists of isolated points with finite geometric multiplicity.

(Kiv) A cumulant, which besides some eigen-operators $A_i^*(\rho)$, $\text{Re}\, x_i > -N$, involves one operator $A_0^*(\rho)$ from the other part of the spectrum, is asymptotically bounded by

$$\left|\left\langle \prod_i A_i^*(S\rho_i) A_0(S\rho_0)\right\rangle_c^*\right| \leq CS^{\Sigma \text{Re}\, x_i - N}. \tag{6.1}$$

If (Kiii) and (Kiv) hold, our results are valid for cumulants of operators with eigenvalues in the half-plane $\text{Re}\, x > -N$, and this may cover all cases of interest.

Evaluating the identities (4.6) and (4.18) we were able to extend the previously established results to operators of arbitrary spatial tensor properties. We furthermore corrected for some inconsistency in the previous results which failed to predict the correct asymptotic behaviour of cumulants involving spatial derivatives. Our discussion has confirmed that conformal covariance provides us with a selection rule which determines those two-point cumulants which show a long-range tail. Furthermore it fixes the asymptotic form of the three-point cumulants. To apply the latter result to a given cumulant $\langle \Pi A_i^*(\rho_i)\rangle_c^*$ we have to know the spectrum of R_L in the half-plane $\mathrm{Re}\, x \geqslant \mathrm{Inf\,Re}\, x_i + 1$.

All these results concern cumulants at the fixed point. An extension of these methods to an arbitrary point on the critical surface is not possible, since the operator K applied to a Hamiltonian $H \neq H^*$ creates non-translational invariant contributions. The corresponding identities only connect $C_{1,\ldots,n}(\rho_1, \ldots, \rho_n, \{g\})$ to quantities without physical interest. Conformal covariance therefore yields useful results only on the leading term of an expansion of $C_{1,\ldots,n}(\rho_1, \ldots, \rho_n, \{g\})$ in powers of $\{g\}$.

We finally want to comment on the special role of the conformal transformation in the context of our treatment. We may set up identities of the type (4.1) for a big variety of functions $u(r)$. The special benefits of the conformal transformation are that it yields operators $K[H^*]$ and $\delta K_L[A_i^*(\rho)]$ which are translational invariant, and which therefore are contained in the initial set of operators. This will not happen in general, and the corresponding identities are useless since they involve unphysical operators.

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Appendix

A.1. Evaluation of $K[H^*]$ and $K_L[A_i^*(\rho)]$

We evaluate $K[H^*]$, using for H^* the explicit form (2.1). This yields (compare equation (3.10)):

$$K[H^*] = \sum_{m=1}^{\infty} \frac{1}{m!} \int \mathrm{d}r_1 \ldots \mathrm{d}r_m \prod_{j=1}^{m} S(r_j) \sum_{i=1}^{m} \left[-\frac{d}{2}\lambda(r_i) + (\nabla_i u(r_i)) + u(r_i)\nabla_i + \lambda(r_i)(b-2\Delta_i) \right]$$
$$\times h^*(r_1, \ldots, r_m) + \sum_{m=0}^{\infty} \frac{1}{m!} \int \mathrm{d}r_1 \ldots \mathrm{d}r_m \prod_{j=1}^{m} S(r_j) \int \mathrm{d}r\, \lambda(r)$$
$$\times (b-2\Delta_x) \left[h^*(x, r_1, \ldots, r_m) - \sum_{n=0}^{m} \binom{m}{n} h^*(r, r_1, \ldots, r_n) h^*(x, r_{n+1}, \ldots, r_m) \right] \Bigg|_{x=r}. \tag{A.1}$$

According to definition (2.6) $R[H^*]$ results from $K[H^*]$ by the substitution $u(r)\to r$, $\lambda(r)\to 1$. We evaluate the mth functional derivative of equation (2.9) which defines H^*:

$$
\begin{aligned}
0=&\left.\frac{\delta^m H^*[S]}{\delta S(r_1)\dots\delta S(r_m)}\right|_{S(r)=0}\\
&=\sum_{i=1}^{m}\left[\frac{d}{2}+r_i\nabla_i+(b-2\Delta_i)\right]h^*(r_1,\dots,r_m)\\
&\quad+\int \mathrm{d}r(b-2\Delta_x)\Big[h^*(x,r,r_1,\dots,r_m)\\
&\quad-\left.\sum_{n=0}^{m}\sum_{p(1,\dots,m;n)}h^*(r,r_{j_1},\dots,r_{j_n})h^*(x,r_{j_{n+1}},\dots,r_{j_m})\Big]\right|_{x=r}.
\end{aligned}
\tag{A.2}
$$

The last sum ranges over all partitionings of the set $(1,\dots,m)$ into subsets $(j_1,\dots,j_n)$ and $(j_{n+1},\dots,j_m)$. We multiply equation (A.2) by

$$
\frac{1}{m!}\prod_{j=1}^{m}S(r_j)\frac{1}{m}\sum_{k=1}^{m}\lambda(r_k),
\tag{A.3}
$$

integrate, and sum over m. The result is subtracted from equation (A.1). This yields

$$
\begin{aligned}
K[H^*]=&\sum_{m=1}^{\infty}\frac{1}{m!}\int \mathrm{d}r_1\dots\mathrm{d}r_m\prod_{j=1}^{m}S(r_j)\Bigg\{\sum_{i=1}^{m}\Big[\nabla_i u(r_i)-\frac{d}{m}\sum_k\lambda(r_k)\\
&+\Big(u(r_i)-r_i\frac{1}{m}\sum\lambda(r_k)\Big)\cdot\nabla_i-2\Big(\lambda(r_i)-\frac{1}{m}\sum\lambda(r_k)\Big)\Delta_i\Big]h^*(r_1,\dots,r_m)\\
&+\int \mathrm{d}r\Big(\lambda(r)-\frac{1}{m}\sum\lambda(r_k)\Big)(b-2\Delta_x)\\
&\times\Big[h^*(r,x,r_1,\dots,r_m)-\sum_{n=0}^{m}\binom{m}{n}h^*(r,r_1,\dots,r_n)\\
&\times h^*(x,r_{n+1},\dots,r_m)\Big]\Big|_{x=r}\Bigg\}+C_K[H^*].
\end{aligned}
\tag{A.4}
$$

The constant $C_k[H^*]$ is of no interest. Note that equation (A.4) holds independently of the special form of $u(r)$ or $\lambda(r)$. In § 5 we therefore take this equation as a starting point to evaluate $K^l[H^*]-K[H^*]$, by substituting $u(r)\to u_l(r,0)-u(r)$, $\lambda(r)\to\lambda_l(r,0)-\lambda(r)$. With the explicit expressions (3.12) and (3.15) some straightforward algebra yields equation (3.18). The contributions with $m=1$ and $m=2$ vanish by virtue of translational, rotational and reflection invariance of H^*.

To evaluate $K_L[A_i^*(\rho)]$ we subtract an expression based on equation (2.17):

$$
\begin{aligned}
&K_L[A_i^*(\rho)]-(x_i\lambda(\rho)-u(\rho)\cdot\nabla_\rho)A_i^*(\rho)\\
&\quad=K_L[A_i^*(\rho)]-\lambda(\rho)R_L^*[A_i^*(\rho)]+(u(\rho)-\lambda(\rho)\rho)\cdot\nabla_\rho A_i^*(\rho).
\end{aligned}
\tag{A.5}
$$

Using the definitions (2.14) and the corresponding expression for K_L it is straightforward to derive equation (3.19). In the course of the calculation there occurs a term

$$2\int dr[(a(r-\rho))\rho - a((r-\rho)\rho)].(\nabla_r S(r))\frac{\delta}{\delta S(r)}A_i^*(\rho)$$

$$=2\sum_{m=1}^{\infty}\frac{1}{m!}\int dr_1\ldots dr_m \prod_{k=1}^{m} S(r_k)\sum_{j=1}^{m}[a((r_j-\rho)\rho)-(a(r_j-\rho))\rho]$$

$$\times\nabla_j a_i^*(\rho, r_1, \ldots, r_m) \tag{A.6}$$

which yields the contribution $\partial\Omega[A_i^*(\rho)]/\partial\epsilon$.

A.2. Evaluation of equation (4.10)

Using the tensor properties of $C_{1,2}^{\{\alpha,\beta\}}(e_0)$ we transform equation (4.10):

$$\frac{\partial}{\partial\epsilon}\sum_{\{\alpha',\beta'\}}\prod_{\mu=1}^{t_1}\Omega_{\alpha_\mu\alpha'_\mu}(\epsilon, e^\perp, e^0)\prod_{\nu=1}^{t_2}\Omega_{\beta_\nu\beta'_\nu}(-\epsilon, e^\perp, e^0)C_{1,2}^{\{\alpha',\beta'\}}(e^0)=0. \tag{A.7}$$

We choose a coordinate system in which the 1-(2-)direction is given by $e^0(e^\perp)$. The change $e^0\to -e^0$ is realized by a rotation $R(\pi)$ of the $(1,2)$-plane with an angle π. Using tensor-space notation we find

$$C_{1,2}(-e_0)=\bigotimes_\mu R^\mu(\pi)\bigotimes_\nu R^\nu(-\pi)C_{1,2}(e_0) \tag{A.8}$$

where we have used $R(\pi)=R(-\pi)$. We represent $R(\pi)$ in the form

$$R(\pi)=\lim_{N\to\infty}\left(I+\frac{\pi}{N}D\right)^N \tag{A.9}$$

$$D=\left(\begin{array}{cc|c} 0 & -1 & \\ 1 & 0 & 0 \\ \hline & 0 & 0 \end{array}\right). \tag{A.10}$$

Some straightforward algebra yields

$$C_{1,2}(-e_0)=\lim_{N\to\infty}\left\{\left\{\bigotimes_\mu I^\mu\bigotimes_\nu I^\nu+\frac{\pi}{N}\left[\sum_{k=1}^{t_1}\left(\bigotimes_{\mu\neq k} I^\mu\otimes D^k\right)\bigotimes_\nu I_\nu\right.\right.\right.$$

$$\left.\left.\left.-\bigotimes_\mu I^\mu\sum_{l=1}^{t_2}\bigotimes_{\nu\neq l} I^\nu\otimes D^l\right]\right\}^N C_{1,2}(e^0)+O\left(\frac{\pi}{N}\right)\right\} \tag{A.11}$$

$$=C_{1,2}(e^0). \tag{A.12}$$

The last result follows by virtue of equation (A.7).

A.3. Examples of three-point cumulants

In solving equation (4.19) we use the *ansatz*

$$C_{1,2,3}^{\{\alpha,\beta,\gamma\}}(\rho_1,\rho_2,\rho_3)=|\rho_{12}|^{\Delta_{12,3}}|\rho_{13}|^{\Delta_{13,2}}|\rho_{23}|^{\Delta_{23,1}}P_{1,2,3}^{\{\alpha,\beta,\gamma\}}(\rho_1,\rho_2,\rho_3). \tag{A.13}$$

We choose $\rho_1=0$. Equation (4.19) reduces to

$$[(a\rho_2^2-2(a\rho_2)\rho_2)\nabla_2+(a\rho_3^2-2(a\rho_3)\rho_3)\nabla_3]P_{1,2,3}^{\{\alpha,\beta,\gamma\}}(\rho_2,\rho_3)$$
$$=\frac{\partial}{\partial\epsilon}\sum_{\{\beta',\gamma'\}}\prod_{\mu=1}^{t_2}\Omega_{\beta_\mu\beta'_\mu}(\epsilon,a,\rho_2)\prod_{\nu=1}^{t_3}\Omega_{\gamma_\nu\gamma'_\nu}(\epsilon,a,\rho_3)P_{1,2,3}^{\{\alpha,\beta',\gamma'\}}(\rho_2,\rho_3). \qquad (A.14)$$

By virtue of the scaling properties $P_{1,2,3}^{\{\alpha,\beta,\gamma\}}(\rho_2,\rho_3)$ depends only on $\rho_2/|\rho_2|$, $\rho_3/|\rho_3|$, and $|\rho_2|/|\rho_3|$. We have solved equation (A.14) for several cases:

(*a*) $t_1=1;\ t_2=t_3=0$

$$P_{1,2,3}^{\alpha}(\rho_1,\rho_2,\rho_3)=\Gamma_{1,2,3}\cdot|\rho_{23}|^{-1}\left(\rho_{21}\frac{|\rho_{31}|}{|\rho_{21}|}-\rho_{31}\frac{|\rho_{21}|}{|\rho_{31}|}\right). \qquad (A.15)$$

(*b*) $t_1=2;\ t_2=t_3=0$

$$P_{1,2,3}^{\alpha_1\alpha_2}(\rho_1,\rho_2,\rho_3)=\Gamma_{1,2,3}\delta_{\alpha_1\alpha_2}. \qquad (A.16)$$

(*c*) $t_1=t_2=1;\ t_3=0$

$$P_{1,2,3}^{\alpha\beta}(\rho_1,\rho_2,\rho_3)=\Gamma_{1,2,3}\left(\delta_{\alpha\beta}-\frac{\rho_{21}^{\alpha}\rho_{21}^{\beta}}{|\rho_{21}|^2}\right). \qquad (A.17)$$

A comparison of expressions (A.16) and (A.17) shows the influence of the rotation Ω in the simplest case.

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